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Abstract: In this paper we study some stability properties of discrete-time systems whose transition map can be approximated by a discrete-time homogeneous transition map. This allows us to identify qualitative stability properties of discrete-time systems by only knowing the discrete-time homogeneity degree of its approximation. We show how these results can be applied to the stability analysis of discrete-time systems obtained by means of the explicit and implicit Euler discretization methods.

Keywords: Nonlinear systems, discrete-time systems, homogeneous systems.

1. INTRODUCTION

A standard technique to simplify the analysis of a nonlinear system is to analyse its local linear approximation. Nevertheless, in many cases, the linear approximation results trivial, singular or unsuitable for analysis or control design (Kawski, 1988).

The concept of weighted homogeneity has allowed the establishment of a wider set of approximating systems: the class of homogeneous systems. Such systems exhibit several interesting features that facilitate the processes of analysis and control design, e.g. scalability of trajectories, finite-time or fixed-time convergence rates, intrinsic robustness to exogenous perturbations and delays (Zubov, 1964; Hahn, 1967; Hermes, 1986; Kawski, 1988; Hermes, 1991; Rosier, 1992; Kawski, 1995; Sepulchre and Aeyels, 1996; Grüne, 2000; Nakamura et al., 2002; Orlov, 2005; Bhat and Bernstein, 2005; Levant, 2005; Andrieu et al., 2008; Nakamura et al., 2009; Bernuau et al., 2013; Sanchez and Moreno, 2017).

For the case of discrete-time systems, the standard definition of weighted homogeneity does not provide, in general, the benefits obtained in the continuous-time case, see e.g. (Hammouri and Benamor, 1999; Tuna and Teel, 2004; Sanchez et al., 2017). For this reason, the concept of $D_{\mathbf{r}}-$homogeneity was introduced in (Sanchez et al., 2017) expressly for discrete-time systems. One of the main properties of $D_{\mathbf{r}}-$homogeneous systems is the simplicity to conclude qualitative stability features directly from the homogeneity degree of the system.

In this paper we define the $D_{\mathbf{r}}-$homogeneous approximation of a discrete-time system. We investigate the conditions that allow us to decide the stability properties of a discrete-time system by means of its $D_{\mathbf{r}}-$homogeneous approximation. We also show how the results can be applied to perform stability analysis of the discrete-time systems obtained by means of the implicit and explicit Euler discretization of continuous-time systems.

Paper organization: In Section 2 the definition and the main properties of $D_{\mathbf{r}}-$homogeneity are recalled. Section 3 contains the results about $D_{\mathbf{r}}-$homogeneous approximation. The application of the results to discretizations of continuous-time systems is given in Section 4. Some useful properties of homogeneous functions are stated in Appendix A. The proofs of the main results are collected in appendices B-D. Several examples about the stability analysis for the discretization of continuous-time systems are provided in Section 5. Some final remarks are stated in Section 6.

Notation: The real and integer numbers are denoted as $\mathbb{R}$ and $\mathbb{Z}$, respectively. $\mathbb{R}_{>0}$ denotes the set $\{x \in \mathbb{R} : x > 0\}$, analogously for the set $\mathbb{Z}$ and the sign $\geq$. For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm and $\|x\|_{\mathbf{r}}$ an $\mathbf{r}-$homogeneous norm (see Definition 1). The composition of two functions f and g (with adequate domains and codomains) is denoted as $f \circ g$, i.e. $(f \circ g)(x) = f(g(x))$. For a continuous positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ and some $\alpha \in \mathbb{R}_{>0}$ we denote

$$\mathcal{I}(V, \alpha) := \{x \in \mathbb{R}^n : V(x) \leq \alpha\},$$
$$\mathcal{E}(V, \alpha) := \{x \in \mathbb{R}^n : V(x) \geq \alpha\}.$$

For $x \in \mathbb{R}$ and $q \in \mathbb{R}_{>0}$, $\lceil x \rfloor^q = \mathrm{sign}(x)|x|^q$.

2. $D_{\mathbf{r}}-$HOMOGENEITY

In this section we recall the concepts of $\mathbf{r}-$homogeneity, $D_{\mathbf{r}}-$homogeneity, and some properties of $D_{\mathbf{r}}-$homogeneous systems. Consider the discrete-time system

$$x(k+1) = f(x(k)), \tag{1}$$

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where the state $x(k) \in \mathbb{R}^n$ for any $k \in \mathbb{Z}_{\geq 0}$. We assume that the transition map $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous for all $x \in \mathbb{R}^n$. Such an assumption guarantees existence and uniqueness of solutions, see e.g. (Agarwal, 2000, p. 5). The solution of (1) with initial condition $x_0 = x(0)$ is denoted as

$$F(k; x_0), \quad \forall\, k \in \mathbb{Z}_{\geq 0}.$$

First, we recall the definition of $\mathbf{r}-$homogeneity.

Definition 1. (Kawski (1988)). Let $\Lambda_\epsilon^{\mathbf{r}}$ denote the family of dilations given by the square diagonal matrix $\Lambda_\epsilon^{\mathbf{r}} = \mathrm{diag}(\epsilon^{r_1}, \ldots, \epsilon^{r_n})$, where $\mathbf{r} = [r_1, \ldots, r_n]^\top$, $r_i \in \mathbb{R}_{>0}$, and $\epsilon \in \mathbb{R}_{>0}$. The components of $\mathbf{r}$ are called the *weights* of the coordinates. Thus:

a) a function $V : \mathbb{R}^n \to \mathbb{R}$ is $\mathbf{r}$-homogeneous of degree $m \in \mathbb{R}$ if $V(\Lambda_\epsilon^{\mathbf{r}} x) = \epsilon^m V(x)$, $\forall x \in \mathbb{R}^n$, $\forall \epsilon \in \mathbb{R}_{>0}$;
b) a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, $f = [f_1, \ldots, f_n]^\top$, is $\mathbf{r}$-homogeneous of degree $\kappa \in \mathbb{R}$ if for each $i = 1, \ldots, n$, $f_i(\Lambda_\epsilon^{\mathbf{r}} x) = \epsilon^{\kappa + r_i} f_i(x)$, $\forall x \in \mathbb{R}^n$, $\forall \epsilon \in \mathbb{R}_{>0}$;
c) given a vector of weights $\mathbf{r}$, a $\mathbf{r}-$homogeneous norm is defined as a function from $\mathbb{R}^n$ to $\mathbb{R}_{\geq 0}$, and given by $\|x\|_{\mathbf{r},p} = \left(\sum_{i=1}^n |x_i|^{p/r_i}\right)^{1/p}$, $\forall x \in \mathbb{R}^n$, for any $p \geq 1$. The set $S_{\mathbf{r}} := \{x \in \mathbb{R}^n : \|x\|_{\mathbf{r},p} = 1\}$ is the corresponding $\mathbf{r}-$homogeneous unit sphere.

Note that any $\mathbf{r}-$homogeneous norm is an $\mathbf{r}-$homogeneous function of degree $m = 1$. Since, for a given $\mathbf{r}$, the $\mathbf{r}-$homogeneous norms are equivalent (Kawski, 1988), they are usually denoted as $\|\cdot\|_{\mathbf{r}}$, with no specification of p.

The definition of $\mathbf{r}$-homogeneity has been particularly useful for analysis and design of continuous-time systems (Bacciotti and Rosier, 2005). However, for discrete-time systems, only the case of $\kappa = 0$ provides clear useful properties (Hammouri and Benamor, 1999; Tuna and Teel, 2004; Sanchez et al., 2017). This situation motivated the introduction of the concept of $D_{\mathbf{r}}-$homogeneity for discrete-time systems.

Definition 2. (Sanchez et al. (2017)). Let $\Lambda_\epsilon^{\mathbf{r}}$, $\mathbf{r}$, and ϵ be as in Definition 1. A map $f : \mathbb{R}^n \to \mathbb{R}^n$, $f = [f_1, \ldots, f_n]^\top$, is $D_{\mathbf{r}}$-homogeneous of degree ν if for each $i = 1, \ldots, n$, $f_i(\Lambda_\epsilon^{\mathbf{r}} x) = \epsilon^{r_i \nu} f_i(x)$, $\forall x \in \mathbb{R}^n$, $\forall \epsilon \in \mathbb{R}_{>0}$, and some $\nu \in \mathbb{R}_{>0}$, or equivalently, $f(\Lambda_\epsilon^{\mathbf{r}} x) = (\Lambda_\epsilon^{\mathbf{r}})^\nu f(x) = \Lambda_\epsilon^{\nu \mathbf{r}} f(x) = \Lambda_{\epsilon^\nu}^{\mathbf{r}} f(x)$.

The system (1) is said to be $D_{\mathbf{r}}-$homogeneous of degree ν if its transition map f is $D_{\mathbf{r}}-$homogeneous of degree ν. The solutions of $D_{\mathbf{r}}-$homogeneous discrete-time systems are interrelated as the solutions of $\mathbf{r}-$homogeneous continuous-time systems (Sanchez et al., 2017).

In order to state the main stability properties of $D_{\mathbf{r}}-$homogeneous systems, let us recall that $x \in \mathbb{R}^n$ is said to be an equilibrium point of (1) if it is a solution of the equation $f(x) - x = 0$.

Theorem 3. (Sanchez et al. (2017)). Suppose that (1) is $D_{\mathbf{r}}-$homogeneous of degree $\nu > 1$. Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuous positive definite $\mathbf{r}$-homogeneous function of degree $m \in \mathbb{R}_{>0}$.

a) If $x = 0$ is an isolated equilibrium point of (1), then it is locally asymptotically stable, and there exists $\alpha \in \mathbb{R}_{>0}$ such that V is a Lyapunov function for (1) on $\mathcal{I}(V, \alpha)$.
b) Suppose that there exists $\beta \in \mathbb{R}_{>0}$ such that $f(x) \neq 0$ and there is no equilibrium of (1) for all $x \in \mathcal{E}(V, \beta)$. Then there exists $\bar{\beta} \in \mathbb{R}_{\geq \beta}$ such that, for all $x_0 \in \mathcal{E}(V, \bar{\beta})$, the solution F of (1) satisfies $|F(k; x_0)| \to \infty$ as $k \to \infty$.

For the next result we recall the definition of ultimate boundedness.

Definition 4. The solutions of (1) are *ultimately bounded* with ultimate bound $\beta \in \mathbb{R}_{>0}$, if for every $\alpha \in \mathbb{R}_{>0}$, there is $T = T(\alpha, \beta) \in \mathbb{Z}_{\geq 0}$, such that

$$|x(0)| \leq \alpha \quad \Rightarrow \quad |x(k)| \leq \beta, \quad \forall k \geq T.$$

Theorem 5. (Sanchez et al. (2017)). Suppose that (1) is $D_{\mathbf{r}}-$homogeneous of degree $\nu \in (0, 1)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous positive definite $\mathbf{r}$-homogeneous function of degree $m \in \mathbb{R}_{>0}$.

a) Suppose that there exists $\alpha \in \mathbb{R}_{>0}$ such that there is no equilibrium of (1) for all $x \in \mathcal{E}(V, \alpha)$. Then the solutions of (1) are globally ultimately bounded, and there exists $\bar{\alpha} \in \mathbb{R}_{\geq \alpha}$ such that $\Delta V(x) := V(f(x)) - V(x) < 0$ for all $x \in \mathcal{E}(V, \bar{\alpha})$.
b) If $x = 0$ is an isolated equilibrium point of (1), then it is locally unstable.

Remark 6. For Theorem 5 point b), the additional condition $f(x) = 0 \Leftrightarrow x = 0$ is asked in (Sanchez et al., 2017). Such a condition is not necessary to prove instability of the origin, however it guarantees that no solution converges to the origin. The proof with the weaker conditions in Theorem 5 is a particular case of the proof of Theorem 10.

In the case $\nu = 0$, $D_{\mathbf{r}}-$homogeneous systems are exponentially converging or diverging, however it depends on the properties of the map f (Sanchez et al., 2017).

3. APPROXIMATION

In this section we consider (1) without assuming that its transition map f is $D_{\mathbf{r}}-$homogeneous. The idea is to verify whether f can be approximated by a $D_{\mathbf{r}}-$homogeneous map h, and whether some stability properties of f can be decided through the properties of h.

First, we give the following definition, which is the discrete-time counterpart of the local (or limit) homogeneity in continuous-time systems.

Definition 7. For a constant $\epsilon_0 \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$, the transition map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be $D_{\mathbf{r}}-$homogeneous of degree $\nu \in \mathbb{R}_{>0}$ in the $(\epsilon_0, h)-$limit, where $h : \mathbb{R}^n \to \mathbb{R}^n$ is some $D_{\mathbf{r}}-$homogeneous map of degree ν, if

$$\lim_{\epsilon \to \epsilon_0} \left(\Lambda_\epsilon^{-\nu \mathbf{r}} f(\Lambda_\epsilon^{\mathbf{r}} x) - h(x)\right) = 0,$$

with the limit computed uniformly for all $x \in S_{\mathbf{r}}$ for $\epsilon_0 \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$.

Before stating the main results of this paper, let us recall that for $\alpha \in \mathbb{R}_{>0}$, $\alpha \in (0, 1]$, a function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be $\alpha-$Hölder continuous in the set $I \subset \mathbb{R}^n$, if there exists $L_I \in \mathbb{R}_{>0}$ such that $|V(x) - V(y)| \leq L_I |x - y|^\alpha$ for all $x, y \in I$, see e.g. (Fiorenza, 2016).

Theorem 8. Suppose that the transition map f of (1) is $D_{\mathbf{r}}-$homogeneous in the $(\epsilon_0, h)-$limit for some $\epsilon_0 \in$

$\{0,+\infty\}$ with some degree $\nu \in \mathbb{R}_{>0}$. Consider a function $V : \mathbb{R}^n \to \mathbb{R}$ being $\alpha-$Hölder continuous in each compact subset of $\mathbb{R}^n$, positive definite, and $\mathbf{r}$-homogeneous of degree $m \in \mathbb{R}_{>0}$.

a) If $\epsilon_0 = 0$, $\nu > 1$, and $x = 0$ is an isolated equilibrium point of h and f, then the origin of (1) is locally asymptotically stable. Moreover, there exists $\gamma \in \mathbb{R}_{>0}$ such that V is a Lyapunov function for (1) on $\mathcal{I}(V,\gamma)$.

b) If $\epsilon_0 = +\infty$, $\nu \in (0,1)$, and there exists $\gamma_0 \in \mathbb{R}_{>0}$ such that $f(x) \neq x$ and $h(x) \neq x$ for all $x \in \mathcal{E}(V,\gamma_0)$, then the solutions of (1) are globally ultimately bounded and there exists $\gamma \geq \gamma_0$ such that $\Delta V(x) < 0$ for all $x \in \mathcal{E}(V,\gamma)$.

Proof. See Appendix B.

Remark 9. A function V with the properties required in Theorem 8 does exist for any vector of weights $\mathbf{r}$, for example $V(x) = \|x\|_{\mathbf{r}}^m$. This is due to any $\mathbf{r}-$homogeneous norm is $\alpha-$Hölder continuous for any $\alpha \in (0,\rho)$ where $\rho^{-1} = \max_{i\in\{1,\dots,n\}}(r_i)$ (Bhat and Bernstein, 2005, Theorem 4.1).

Theorem 10. Suppose that f and V are as in Theorem 8.

a) If $\epsilon_0 = +\infty$, $\nu > 1$, and there exists $\gamma_0 \in \mathbb{R}_{>0}$ such that $f(x) \neq 0$, $f(x) \neq x$, $h(x) \neq 0$ and $h(x) \neq x$ for all $x \in \mathcal{E}(V,\gamma_0)$, then there exists $\gamma_1 \geq \gamma_0$ such that $|F(k;x_0)| \to \infty$ as $k \to \infty$ for all $x_0 \in \mathcal{E}(V,\gamma_1)$.

b) If $\epsilon_0 = 0$, $\nu \in (0,1)$, $f(0) = 0$, $h(0) = 0$, and there exists $\gamma_0 \in \mathbb{R}_{>0}$ such that $f(x) \neq 0$, $f(x) \neq x$, $h(x) \neq x$ for all $x \in \mathcal{I}(V,\gamma_0) \setminus \{0\}$, then the origin of (1) is locally unstable.

Proof. See Appendix C.

4. APPLICATION TO THE ANALYSIS OF DISCRETIZED CONTINUOUS-TIME SYSTEMS

In this section we consider the following continuous-time system

$$\dot{x}(t) = g(x(t)), \quad x(t) \in \mathbb{R}^n, \tag{2}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field. If (2) is wanted to be numerically solved, then a discretization method is required. Two of the simplest ones are the Euler methods: the explicit (or forward) and the implicit (or backward), see e.g. (Hairer et al., 1993, Section II.7). Below we recall such methods, and study some stability properties of the discrete-time system obtained by their application to (2).

4.1 Explicit Euler method

The explicit Euler discretization (EED) of (2), with a step $\tau \in \mathbb{R}_{>0}$, is given by (see e.g. (Hairer et al., 1993)) $x((k+1)\tau) - x(k\tau) = \tau g(x(k\tau))$. Thus, the EED of (2) is given by the discrete-time system

$$x((k+1)\tau) = G(x(k\tau)), \tag{3}$$

where the map $G : \mathbb{R}^n \to \mathbb{R}^n$ is given by $G(y) = y + \tau g(y)$.

From Theorem 10 we can immediately deduce the following properties of (3).

Corollary 11. Consider (2) and its EED (3). Suppose that g is $D_{\mathbf{r}}-$homogeneous in the $(\epsilon_0, H)-$limit for some $\epsilon_0 \in \{0,+\infty\}$ with some degree $\nu \in \mathbb{R}_{>0}$. Let V be as in Theorem 8.

a) If $\epsilon_0 = +\infty$, $\nu > 1$, and there exists $\gamma_0 \in \mathbb{R}_{>0}$ such that $\tau g(x) \neq -x$, $g(x) \neq 0$, $H(x) \neq 0$ and $\tau H(x) \neq x$ for all $x \in \mathcal{E}(V,\gamma_0)$, then there exists $\gamma_1 \geq \gamma_0$ such that the solution F of (3) satisfies $|F(k;x_0)| \to \infty$ as $k \to \infty$ for all $x_0 \in \mathcal{E}(V,\gamma_1)$.

b) If $\epsilon_0 = 0$, $\nu \in (0,1)$, $g(0) = 0$, $H(0) = 0$, and there exists $\gamma_0 \in \mathbb{R}_{>0}$ such that $\tau g(x) \neq -x$, $g(x) \neq 0$ and $\tau H(x) \neq x$ for all $x \in \mathcal{I}(V,\gamma_0) \setminus \{0\}$, then the origin of (3) is locally unstable.

Observe that the conditions in Corollary 11 do not consider the map G, but only the vector field g. This is clear by noting that for the identity map $I(x) = x$, we have that $\Lambda_\epsilon^{-\nu\mathbf{r}} I(\Lambda_\epsilon^{\mathbf{r}} x) = \Lambda_\epsilon^{(1-\nu)\mathbf{r}} x$ thus:

- for $\nu \in (0,1)$, $\Lambda_\epsilon^{(1-\nu)\mathbf{r}} x \to 0$ as $\epsilon \to 0$, and;
- for $\nu > 1$, $\Lambda_\epsilon^{(1-\nu)\mathbf{r}} x \to 0$ as $\epsilon \to +\infty$.

Remark 12. Observe that the instability features of a EED concluded in Corollary 11 only depend on the homogeneity degree of the vector field g and not on the stability properties of g. Thus, Corollary 11 is useful to detect inconsistencies in the EED of a continuous-time system whose vector field has a $D_{\mathbf{r}}-$homogeneous approximation. This fact is clarified in Section 5.1 where the origin of a continuous-time system is asymptotically stable but the origin of its EED is unstable. Hence, it is clear the need for discretization schemes that guarantee the preservation of the stability features of the continuous-time systems they are discretizing.

Now, although less possible due to the linear term in G, the following result can be stated from Theorem 8.

Corollary 13. Consider (2) and its EED (3). Let V be as in Theorem 8.

a) If for some $\tau = \tau^* \in \mathbb{R}_{>0}$, G is $D_{\mathbf{r}}-$homogeneous of degree $\nu \in (0,1)$ in the $(+\infty, H)-$limit, and there exists $\gamma_0 \in \mathbb{R}_{>0}$ such that $G(x) \neq x$ and $H(x) \neq x$ for all $x \in \mathcal{E}(V,\gamma_0)$, then for $\tau = \tau^*$ the solutions of (3) are globally ultimately bounded and there exists $\gamma \geq \gamma_0$ such that $\Delta V < 0$ for all $x \in \mathcal{E}(V,\gamma)$.

b) If for some $\tau = \tau^* \in \mathbb{R}_{>0}$, G is $D_{\mathbf{r}}-$homogeneous of degree $\nu > 1$ in the $(0,H)-$limit, and $x = 0$ is an isolated equilibrium point of G and H, then for $\tau = \tau^*$ the origin of (3) is locally asymptotically stable. Moreover, there exists $\gamma \in \mathbb{R}_{>0}$ such that V is a Lyapunov function for (3) on $\mathcal{I}(V,\gamma)$.

4.2 Implicit Euler method

The implicit Euler discretization (IED) of (2), with a step $\tau \in \mathbb{R}_{>0}$, is given by $x((k+1)\tau) - x(k\tau) = \tau g(x((k+1)\tau))$, or equivalently

$$x(k\tau) = x((k+1)\tau) - \tau g(x((k+1)\tau)).$$

Let us define the map $G : \mathbb{R}^n \to \mathbb{R}^n$ given by $G(y) = y - \tau g(y)$ and assume that G is invertible with inverse G^{-1}. Thus, the IED of (2) is given by the discrete-time system

$$x((k+1)\tau) = G^{-1}(x(k\tau)). \tag{4}$$

As it was done for the explicit Euler discretization, we can use $D_{\mathbf{r}}-$homogeneous approximations to analyse (4). But let us first state the following auxiliary lemma.

Lemma 14. Consider an invertible map $g:\mathbb{R}^n\to\mathbb{R}^n$, and an invertible $D_{\mathbf{r}}-$homogeneous map $h:\mathbb{R}^n\to\mathbb{R}^n$ of some degree $\nu\in\mathbb{R}_{>0}$.

(1) The map h^{-1} is $D_{\mathbf{r}}-$homogeneous of degree ν^{-1}.
(2) If g is $D_{\mathbf{r}}-$homogeneous in the $(+\infty,h)-$limit (in the $(0,h)-$limit, respectively), then the map $g^{-1}:\mathbb{R}^n\to\mathbb{R}^n$ is $D_{\mathbf{r}}-$homogeneous in the $(+\infty,h^{-1})-$limit (in the $(0,h^{-1})-$limit, respectively).

Proof. See Appendix D.

Corollary 15. Consider (2) and its IED (4). Suppose that g is $D_{\mathbf{r}}-$homogeneous in the $(\epsilon_0,H)-$limit for some $\epsilon_0\in\{0,+\infty\}$ with some degree $\nu\in\mathbb{R}_{>0}$. Suppose that G and H are invertible and let V be as in Theorem 8.

a) If $\epsilon_0=0$, $\nu\in(0,1)$, $g(0)=0$, $H(0)=0$, and there exists $\gamma_0\in\mathbb{R}_{>0}$ such that $\tau H(x)\neq x$, $g(x)\neq 0$, for all $x\in\mathcal{I}(V,\gamma_0)\setminus 0$, then the origin of (4) is locally asymptotically stable. Moreover, there exists $\gamma\le\gamma_0$ such that V is a Lyapunov function for (4) in $\mathcal{I}(V,\gamma)$.
b) If $\epsilon_0=+\infty$, $\nu>1$, and there exists $\gamma_0\in\mathbb{R}_{>0}$ such that $g(x)\neq 0$ and $\tau H(x)\neq x$ for all $x\in\mathcal{E}(V,\gamma_0)$, then the solutions of (4) are globally ultimately bounded, and there exists $\gamma\ge\gamma_0$ such that $\Delta V(x)<0$ for all $x\in\mathcal{E}(V,\gamma)$.

Remark 16. Note that (as it was explained in Remark 12), Corollary 15 is useful to detect inconsistencies in the IED of a continuous-time system whose vector field has a $D_{\mathbf{r}}-$homogeneous approximation. In Section 5.2 it is shown a continuous-time system whose origin is globally unstable, however, the origin of its IED is locally asymptotically stable.

Corollary 17. Consider (2) and its IED (4). Suppose that G is $D_{\mathbf{r}}-$homogeneous in the $(\epsilon_0,H)-$limit for some $\epsilon_0\in\{0,+\infty\}$ with some degree $\nu\in\mathbb{R}_{>0}$, for some $\tau=\tau^*\in\mathbb{R}_{>0}$. Suppose that G and H are invertible and let V be as in Theorem 8.

a) If $\epsilon_0=+\infty$, $\nu\in(0,1)$, and there exists $\gamma_0\in\mathbb{R}_{>0}$ such that $G(x)\neq 0$, $H(x)\neq 0$, $G(x)\neq x$ and $H(x)\neq x$ for all $x\in\mathcal{E}(V,\gamma_0)$, then for $\tau=\tau^*$ there exists $\gamma\ge\gamma_0$ such that the solution F of (4) satisfies $|F(k;x_0)|\to\infty$ as $k\to\infty$ for all $x_0\in\mathcal{E}(V,\gamma)$.
b) If $\epsilon_0=0$, $\nu>1$, and there exists $\gamma_0\in\mathbb{R}_{>0}$ such that $x=0$ is a unique equilibrium point of G and H in $\mathcal{I}(V,\gamma_0)$, then for $\tau=\tau^*\in\mathbb{R}_{>0}$, the origin of (4) is locally unstable.

The results of this section are in accordance to (Efimov et al., 2017), where a thorough stability analysis of IED and EED for continuous-time $\mathbf{r}-$homogeneous systems is presented (this fact is illustrated in Section 5.4). Nonetheless, the advantage of $D_{\mathbf{r}}-$homogeneous approximation lies in the facility to verify stability properties by considering only the homogeneity degree and not needing information about the Lyapunov function of the continuous-time system.

5. EXAMPLES

In this section we exemplify the results obtained in the previous sections. Nonetheless, in sections 5.1 and 5.2 we provide some simple examples that can be solved analytically to verify the results obtained in Section 4.

5.1 Two scalar systems I

a) Consider the following continuous-time scalar system

$$\dot{x}(t)=-\lceil x(t)\rfloor^{1/2},\quad x(t)\in\mathbb{R}. \tag{5}$$

Let us stress that, the origin of (5) is asymptotically stable. The EED of (5) is given by

$$x((k+1)\tau)=x(k\tau)-\tau\lceil x(k\tau)\rfloor^{1/2}. \tag{6}$$

Note that the transition map of (6) is $D_{\mathbf{r}}-$homogeneous of degree $\nu=1/2$ in the $(0,h)-$limit with $h(x)=-\tau\lceil x\rfloor^{1/2}$.

Now, if $x(k\tau)\in(0,\tau^2/4)$, then $x((k+1)\tau)<0$. Moreover, $-x((k+1)\tau)-x(k\tau)=-2x(k\tau)+\tau\sqrt{x(k\tau)}>0$. Analogously, for $x(k\tau)\in(-\tau^2/4,0)$, we have that $x((k+1)\tau)>0$, and $x((k+1)\tau)+x(k\tau)>0$. Hence, the origin of (6) is unstable and the instability domain is the interval $(-\tau^2/4,\tau^2/4)$.

b) On the other hand, the origin of the continuous-time system

$$\dot{x}(t)=-\lceil x(t)\rfloor^{2},\quad x(t)\in\mathbb{R}, \tag{7}$$

is globally asymptotically stable. Its EED is given by

$$x((k+1)\tau)=x(k\tau)-\tau\lceil x(k\tau)\rfloor^{2}. \tag{8}$$

Note that the transition map of (8) is $D_{\mathbf{r}}-$homogeneous of degree $\nu=2$ in the $(+\infty,h)-$limit with $h(x)=-\tau\lceil x\rfloor^2$. It is easy to verify analytically that, for any initial condition $x(0)\notin[-2/\tau,2/\tau]$ the solutions of (8) diverge.

5.2 Two scalar systems II

a) Consider the following continuous-time scalar system

$$\dot{x}(t)=\lceil x(t)\rfloor^{1/2},\quad x(t)\in\mathbb{R}. \tag{9}$$

The origin of this system is globally unstable. The IED of (9) is given by

$$x(k\tau)=x((k+1)\tau)-\tau\lceil x((k+1)\tau)\rfloor^{1/2}. \tag{10}$$

Note that the right hand side of (10) is $D_{\mathbf{r}}-$homogeneous of degree $\nu=1/2$ in the $(0,h)-$limit with $h(x)=-\tau\lceil x\rfloor^{1/2}$. The map $f(y)=y-\tau\lceil y\rfloor^{1/2}$ is invertible in the neighbourhood of the origin given by $(-\tau^2/4,\tau^2/4)$. Thus, for such a neighbourhood the explicit representation associated to (10) is given by

$$x((k+1)\tau)=\begin{cases} s_1(x(k\tau)), & x(k\tau)\in(-\tau^2/4,0),\\ s_2(x(k\tau)), & x(k\tau)\in[0,\tau^2/4),\end{cases} \tag{11}$$

where

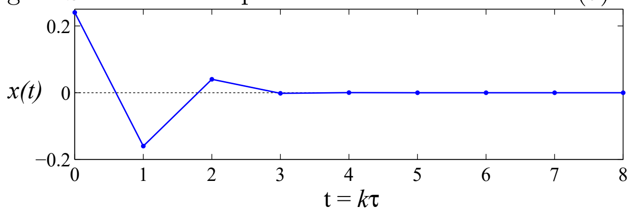
$$s_1(x(k\tau))=x(k\tau)+\tau^2/2-\tau\sqrt{x(k\tau)+\tau^2/4},$$
$$s_2(x(k\tau))=x(k\tau)-\tau^2/2+\tau\sqrt{-x(k\tau)+\tau^2/4}.$$

To verify the asymptotic stability of the origin of (11) consider $x(k\tau)=\tau^2/4-\epsilon$ with $\epsilon\in(0,\tau^2/4)$. Thus, by substituting in (11), we obtain

$$x((k+1)\tau)=-\tau^2/4-\epsilon+\tau\sqrt{\epsilon}. \tag{12}$$

Note that $x((k+1)\tau)<0$, thus, we have to verify that $-x((k+1)\tau)<x(k\tau)$. This is true if and only if

Fig. 1. State of the implicit Euler discretization of (9).



$2\epsilon < \tau\sqrt{\epsilon}$, and this is the case for any $\epsilon \in (0, \tau^2/4)$. An analogous computation holds for $x(k\tau) = -\tau^2/4 + \epsilon$. This proves that the norm of the solution of (11) is a strictly decreasing function of k for any initial condition $x(0) \in (-\tau^2/4, \tau^2/4) \setminus \{0\}$.

In Fig. 1 we can see the simulation of (11) with $\tau = 1$ and initial condition $x(0) = 0.24$.

b) Now consider the continuous-time system

$$\dot{x}(t) = \lceil x(t) \rfloor^2 , \quad x(t) \in \mathbb{R} , \tag{13}$$

whose origin is globally unstable. Its IED is given by

$$x(k\tau) = x((k+1)\tau) - \tau \lceil x((k+1)\tau) \rfloor^2 . \tag{14}$$

Note that the right hand side of (14) is $D_{\mathbf{r}}-$homogeneous of degree $\nu = 2$ in the $(+\infty, h)-$limit with $h(x) = -\tau\lceil x \rfloor^2$, and it is invertible outside the interval $[-2/\tau, 2/\tau]$. Thus, the explicit representation of (14) is given by

$$x((k+1)\tau) = \begin{cases} s_1(x(k\tau)), & x(k\tau) < -2/\tau , \\ s_2(x(k\tau)), & x(k\tau) > 2/\tau , \end{cases} \tag{15}$$

where

$$s_1(x(k\tau)) = 1/(2\tau) + \sqrt{-x(k\tau)/\tau + 1/(4\tau^2)} ,$$
$$s_2(x(k\tau)) = 1/(2\tau) - \sqrt{x(k\tau)/\tau + 1/(4\tau^2)} .$$

By an analogous analysis as in a), it can be verify that, for any initial condition $x(0) \notin [-2/\tau, 2/\tau]$ the solutions of (15) are ultimately bounded with final bound $2/\tau$.

5.3 Duffing's equation

In this example we consider the Duffing's equation given by (see e.g. (Strogatz, 1994)) $\ddot{z} + z + dz^3 = 0$, $d \in \mathbb{R}_{>0}$, whose state space representation with $x_1 = z$ and $x_2 = \dot{x}$ is given by

$$\dot{x}_1 = x_2 , \quad \dot{x}_2 = -x_1 - dx_1^3 . \tag{16}$$

It is important to mention that (16) is an oscillator, thus its solutions are bounded for any initial condition.

Note that the map g given by $g(x) = [x_2, -x_1 - dx_1^3]^\top$ is $D_{\mathbf{r}}-$homogeneous in the $(0, H)-$limit with $\nu = \sqrt{3}$, $\mathbf{r} = [1, \sqrt{3}]^\top$, and H given by $H(x) = [x_2, -dx_1^3]^\top$.

According to Corollary 11 point a), for any $\tau \in \mathbb{R}_{>0}$ there is a neighbourhood of the origin such that the solutions of the EED of (16) are unbounded for initial conditions outside of such a neighbourhood. For the simulations, we set the parameter $d = 1/2$ and the integration step $\tau = 0.1$. Fig. 5 shows a simulation of the EED of (16) with initial conditions $x_1(0) = 0.1$, $x_2(0) = 0$.

Now, Corollary 15 point b) guarantees that the solutions of the IED of (16) are globally ultimately bounded for any $\tau \in \mathbb{R}_{>0}$. Fig 3 shows a simulation of the IED of (16) for the initial conditions $x_1(0) = 10$, $x_2(0) = 0$.

Fig. 2. States of the explicit Euler discretization of (16).

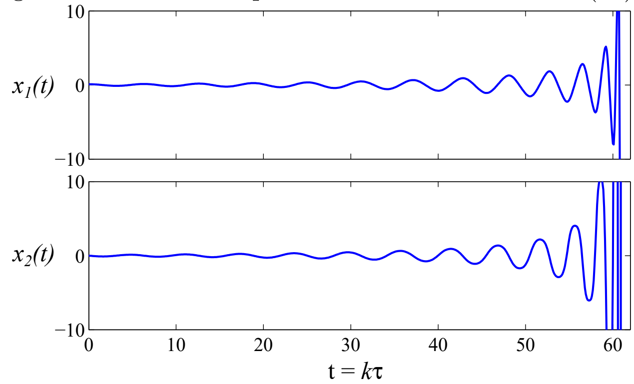
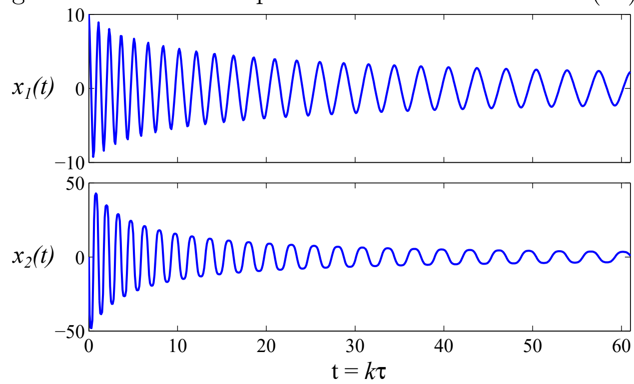


Fig. 3. States of the implicit Euler discretization of (16).



5.4 Homogeneous control of the double integrator

In this section we consider the following controlled system

$$\dot{x}_1 = x_2 , \quad \dot{x}_2 = u(x)\,._i , \quad i \in \{1, 2, 3\} , \tag{17}$$

We analyse two different cases for the feedback controller $u(x)$, namely $u \in \{u_1, u_2\}$ where

$$u_1(x) := -a_1\lceil x_1 \rfloor^{1/3} - a_2\lceil x_2 \rfloor^{1/2} ,$$
$$u_2(x) := -b_1\lceil x_1 \rfloor^{3} - b_2\lceil x_2 \rfloor^{3/2} .$$

Case: u_1. Observe that the closed-loop of (17) with u_1 is $\bar{\mathbf{r}}-$homogeneous of degree $\kappa = -1$ with $\bar{\mathbf{r}} = [3, 2]^\top$. Moreover, its origin is globally finite-time stable (Bhat and Bernstein, 1997; Orlov et al., 2011), for all $a_1, a_2 \in \mathbb{R}_{>0}$ (Bernuau et al., 2015).

Note that the map g given by $g(x) = [x_2, u_1(x)]^\top$ is $D_{\mathbf{r}}-$homogeneous in the $(0, H)-$limit with $\nu = 1/2$, $\mathbf{r} = [2, 1]^\top$, and the map H is given by $H(x) = [x_2, -a_2\lceil x_2 \rfloor^{1/2}]^\top$. Thus, according to Corollary 11 point b), the origin of the EED of (17) with u_1 is unstable for any $\tau \in \mathbb{R}_{>0}$. For the simulation we use the parameters: $a_1 = 10$, $a_2 = 5$, and $\tau = 0.2$. Fig. 4 shows the instability of the origin of the EED of (17) with u_1 and the initial conditions $x_1(0) = 0.01$, $x_2(0) = 0.01$. Observe that in this case H is not invertible, then we cannot use Corollary 15 to study the IED of (17) in closed-loop with u_1.

Case: u_2. Now, (17) in closed-loop with u_2 is $\bar{\mathbf{r}}-$homogeneous of degree $\kappa = 1$ with $\bar{\mathbf{r}} = [1, 2]^\top$.

Note that the map g given by $g(x) = [x_2, u_2(x)]^\top$ is $D_{\mathbf{r}}-$homogeneous in the $(0, H)-$limit with $\nu = \sqrt{3}$, $\mathbf{r} = [1, \sqrt{3}]^\top$, and H given by $H(x) = [x_2, -b_1\lceil x_1 \rfloor^3]^\top$. In this

Fig. 4. States of the explicit Euler discretization of (17) with u_1.

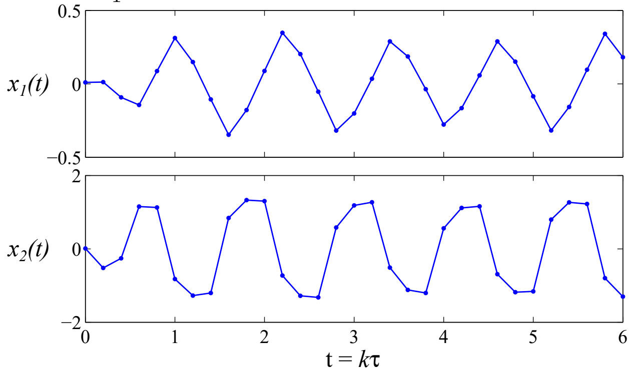
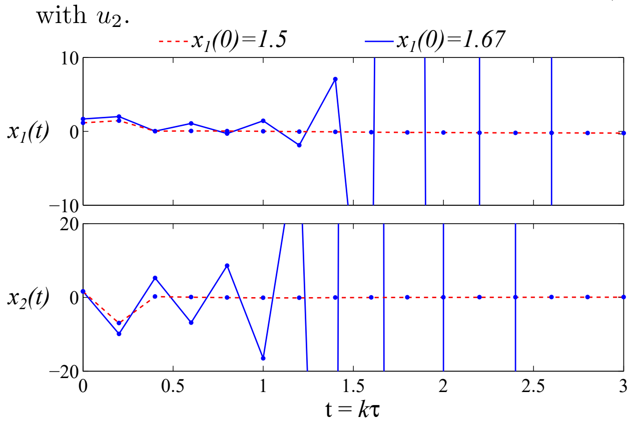


Fig. 5. States of the explicit Euler discretization of (17) with u_2.



example, we consider the parameters $b_1 = 10$ and $b_2 = 5$. For such a case, the origin of (17) in closed-loop with u_2 is globally asymptotically stable (Efimov et al., 2017).

According to Corollary 11 point a), for any $\tau \in \mathbb{R}_{>0}$ there is a neighbourhood of the origin such that the solutions of the EED of (17) with u_2 are unbounded for all initial conditions outside such a neighbourhood. This situation is shown in Fig. 5, where the integration step is $\tau = 0.2$. For the initial conditions $x_1(0) = 1.5$, $x_2(0) = 1.5$, the states converge to the origin, but by increasing the initial condition for x_1 to $x_1(0) = 1.67$ the system's states become unbounded.

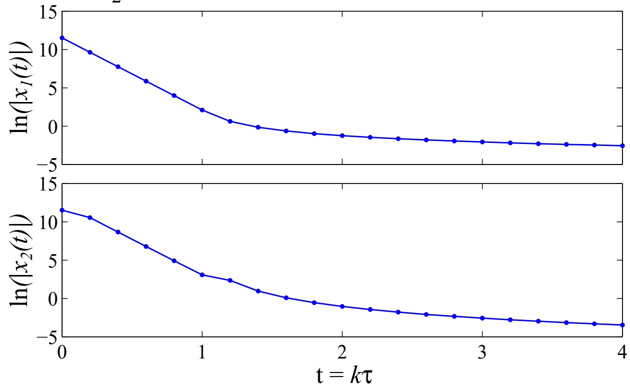
On the other hand, Corollary 15 point b) guarantees that the solutions of the IED of (17) with u_2 are globally ultimately bounded for any $\tau \in \mathbb{R}_{>0}$. Fig. 6 shows the states of the IED of (17) with u_2 and the initial conditions $x_1(0) = 1 \times 10^5$, $x_2(0) = 1 \times 10^5$.

6. CONCLUSION

In this paper we have provided a methodology to study some stability properties of discrete-time systems by means of $D_{\mathbf{r}}-$homogeneous approximations.

The qualitative stability features of a system can be decided in a simple way. However, to obtain quantitative estimates (e.g. size of attraction domains) a more detailed analysis is required. Nevertheless, $D_{\mathbf{r}}-$homogeneity guarantees the existence of Lyapunov functions that can be used for such a purpose.

Fig. 6. States of the implicit Euler discretization of (17) with u_2.



We have also shown that the presented methodology can be used to provide criteria to choose suitable discretization techniques for continuous-time systems.

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Appendix A. HOMOGENEOUS FUNCTIONS

We state some useful properties of $\mathbf{r}-$homogeneous functions.

Lemma 18. (Bhat and Bernstein (2005)). Suppose that the functions $V_1, V_2 : \mathbb{R}^n \to \mathbb{R}$ are continuous, $\mathbf{r}$-homogeneous of degrees $m_1, m_2 \in \mathbb{R}_{>0}$, respectively, and V_1 is positive definite. Then,

$$\underline{\gamma} V_1^{\frac{m_2}{m_1}}(x) \leq V_2(x) \leq \bar{\gamma} V_1^{\frac{m_2}{m_1}}(x)\,,$$

for every $x \in \mathbb{R}^n$, where $\underline{\gamma} = \min_{x\in E} V_2(x)$, and $\bar{\gamma} = \max_{x\in E} V_2(x)$, with $E = \{x \in \mathbb{R}^3 : V_1(x) = 1\}$.

Lemma 19. (Sanchez et al. (2017)). Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous $\mathbf{r}-$homogeneous function of degree m. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a $D_\mathbf{r}-$homogeneous map of degree ν. Then: a) $V \circ f$ is an $\mathbf{r}-$homogeneous function of degree $\bar{m} = \nu m$; b) $V \circ f$ is locally bounded if f is locally bounded, and $V \circ f$ is continuous if f is continuous; c) $V \circ f$ is positive semidefinite if V is positive semidefinite; d) $V \circ f$ is positive definite if V is positive definite and f is such that $f(x) = 0$ if and only if $x = 0$.

Appendix B. PROOF OF THEOREM 8

The idea of the proof is the following: since V is useful to verify the stability properties of the transition map h, we will use V to verify the same stability properties of f. Along (1), $\Delta V(x) = V(f(x)) - V(x)$ can be rewritten as $\Delta V(x) = V(h(x)) - V(x) + V_d(x)$ where $V_d(x) = V(f(x)) - V(h(x))$.

Now, let us analyse the terms $V(h(x))$ and $V_d(x)$ to subsequently find an upper bound for ΔV. According to lemmas 18 and 19 given in Appendix A, there exists $\bar{\gamma} \in \mathbb{R}_{>0}$ such that $V(h(x)) \leq \bar{\gamma} V^\nu(x)$. Thus, $V(h(x)) - V(x) \leq -(1 - \bar{\gamma} V^{\nu-1}(x))V(x)$. Hence, it is easy to see that for $\nu > 1$ (respectively, for $\nu \in (0, 1)$) there exist $\gamma_0, \gamma_1 \in \mathbb{R}_{>0}$ such that $V(h(x)) - V(x) \leq -\gamma_1 V(x)$ for all $x \in \mathcal{I}(V, \gamma_0)$ (respectively, for all $x \in \mathcal{E}(V, \gamma_0)$) (Sanchez et al., 2017).

For the analysis of $V_d(x)$, define $\bar{x} = \Lambda^\mathbf{r}_{\|x\|_\mathbf{r}^{-1}} x$ for all $x \in \mathcal{I}(V, \gamma_0)$, $x \neq 0$ (respectively, for all $x \in \mathcal{E}(V, \gamma_0)$). Observe that $\|\bar{x}\|_\mathbf{r} = 1$. Denote $y_{\|x\|_\mathbf{r}}(\bar{x}) = \Lambda^{-\nu\mathbf{r}}_{\|x\|_\mathbf{r}} f(\Lambda^\mathbf{r}_{\|x\|_\mathbf{r}} \bar{x})$, thus, $V_d(x) = V(f(\Lambda^\mathbf{r}_{\|x\|_\mathbf{r}} \bar{x})) - V(h(\Lambda^\mathbf{r}_{\|x\|_\mathbf{r}} \bar{x})) = V(f(\Lambda^\mathbf{r}_{\|x\|_\mathbf{r}} \bar{x})) - V(\Lambda^{\nu\mathbf{r}}_{\|x\|_\mathbf{r}} h(\bar{x})) = \|x\|_\mathbf{r}^{\nu m} \left[V(y_{\|x\|_\mathbf{r}}(\bar{x})) - V(h(\bar{x}))\right]$. Since $y_{\|x\|_\mathbf{r}}$ and h are continuous maps, there exists a compact set $C \subset \mathbb{R}^n$ such that $y_{\|x\|_\mathbf{r}}(\bar{x}), h(\bar{x}) \in C$ for all $\bar{x} \in S_\mathbf{r}$. Hence, by $\alpha-$Hölder continuity of V, there exists $L_C \in \mathbb{R}_{>0}$ such that $|V_d(x)| \leq \|x\|_\mathbf{r}^{\nu m} L_C |y_{\|x\|_\mathbf{r}}(\bar{x}) - h(\bar{x})|^\alpha$

for all $x \in \mathcal{I}(V, \gamma_0)$, (respectively, for all $x \in \mathcal{E}(V, \gamma_0)$). Note that, for some $\gamma_2 \in \mathbb{R}_{>0}$, we have that $|V_d(x)| \leq \gamma_2 V^{\nu}(x) L_C |y_{\|x\|_{\mathbf{r}}}(\bar{x}) - h(\bar{x})|^{\alpha}$. Thus

$$\Delta V(x) \leq -\left(\gamma_1 - \gamma_2 V^{\nu-1}(x) L_C |y_{\|x\|_{\mathbf{r}}}(\bar{x}) - h(\bar{x})|^{\alpha}\right) V(x).$$

Since f is $D_{\mathbf{r}}-$homogeneous of degree $\nu > 1$ in the $(0, h)-$limit (respectively, of degree $\nu \in (0,1)$ in the $(+\infty, h)-$limit), $y_{\|x\|_{\mathbf{r}}}(\bar{x}) - h(\bar{x}) \to 0$ as $\|x\|_{\mathbf{r}} \to 0$ (respectively, as $\|x\|_{\mathbf{r}} \to +\infty$), see Definition 7. Hence, there exists $\gamma \in \mathbb{R}_{>0}$ such that $\gamma \leq \gamma_0$ (respectively, $\gamma \geq \gamma_0$), and $\gamma_1 > \gamma_2 V^{\nu-1}(x) L_C |y_{\|x\|_{\mathbf{r}}}(\bar{x}) - h(\bar{x})|^{\alpha}$, for all $x \in \mathcal{I}(V, \gamma)$ (respectively, for all $x \in \mathcal{E}(V, \gamma)$).

Therefore, if $\nu > 1$, then V is a Lyapunov function for (1), which proves local asymptotic stability. For the case $\nu \in (0,1)$, Corollary 5.14.3 in (Agarwal, 2000) guarantees the existence of T required in Definition 4.

Appendix C. PROOF OF THEOREM 10

a) To prove this point, we will show the existence of γ_1 such that, along (1), $\Delta V(x)$ is positive for all $x \in \mathcal{E}(V, \gamma_1)$.

Firstly, note that lemmas 18 and 19 given in Appendix A ensure the existence of $\gamma_2 \in \mathbb{R}_{>0}$ such that $V(h(x)) \geq \gamma_2 V^{\nu}(x)$ for all $x \in \mathbb{R}^n$. Thus, $\Delta V(x) \geq \gamma_2 V^{\nu}(x) - V(x) - |V_d(x)| \geq \gamma_2 V^{\nu}(x) - V(x) - V^{\nu}(x) L_C |y_{\|x\|_{\mathbf{r}}}(\bar{x}) - h(\bar{x})|^{\alpha}$, where V_d, L_C, $y_{\|x\|_{\mathbf{r}}}$, and $\bar{x}$ are as defined in Appendix B. Hence,

$$\Delta V(x) \geq V^{\nu}(x)\left(\gamma_2 - V^{1-\nu}(x) - L_C |y_{\|x\|_{\mathbf{r}}}(\bar{x}) - h(\bar{x})|^{\alpha}\right).$$

Since f is $D_{\mathbf{r}}-$homogeneous of degree $\nu > 1$ in the $(+\infty, h)-$limit, $y_{\|x\|_{\mathbf{r}}}(\bar{x}) - h(\bar{x}) \to 0$ as $\|x\|_{\mathbf{r}} \to +\infty$. Therefore, there exist $\gamma_1, \gamma_3 \in \mathbb{R}_{>0}$ such that $\gamma_2 - V^{1-\nu}(x) - L_{C_1} |y_{\epsilon}(\bar{x}) - h(\bar{x})|^{\alpha} \geq \gamma_3$ for all $x \in \mathcal{E}(V, \gamma_1)$.

Now, we want to verify that the trajectories with initial conditions in $\mathcal{E}(V, \gamma_1)$ diverge. For all $x \in \mathcal{E}(V, \gamma_1)$, we have obtained the inequality $\Delta V(x) \geq \gamma_3 V^{\nu}(x)$ which implies that $V(F(k+1; x_0)) > V(F(k; x_0))$ for all $k \in \mathbb{Z}_{\geq 0}$. Moreover, $\sum_{j=0}^{k} \Delta V(x(j)) = V(F(k; x_0)) - V(x_0)$, and $\sum_{j=0}^{k} \Delta V(x(j)) \geq \sum_{j=0}^{k} \gamma_3 V^{\nu}(x(j)) \geq k\gamma_3 V^{\nu}(x(0))$. Thus, $V(F(k; x_0)) \geq V(x_0) + k\gamma_3 V^{\nu}(x(0))$ for all $k \in \mathbb{Z}_{\geq 0}$. From this inequality it is clear that

$$k \to \infty \;\Rightarrow\; V(F(k; x_0)) \to \infty \;\Rightarrow\; |F(k; x_0)| \to \infty.$$

b) The proof of this point follows the same ideas of the Chetaev's instability theorem for continuous-time systems, see also (Agarwal, 2000, Theorem 5.10.4).

First, we construct a nonempty open set whose boundary contains the origin. Since $x = 0$ is an isolated equilibrium point of h, there exists $\gamma_0 \in \mathbb{R}_{>0}$ such that $x = 0$ is the unique equilibrium point for all $x \in \mathcal{I}(V, \gamma_0)$. Now, since h is a nontrivial map, there exists $y \in \mathcal{I}(V, \gamma_0)$ such that $h(y) \neq 0$. Thus, by $D_{\mathbf{r}}$-homogeneity of h we have that $h(\Lambda_{\epsilon}^{\mathbf{r}} y) \neq 0$ for all $\epsilon \in \mathbb{R}_{>0}$. Consider the sets $S_{\gamma_1} = \{x \in \mathbb{R}^n : V(x) = \gamma_1\}$ for any $\gamma_1 \in (0, \gamma_0)$. Since any S_{γ_1} is compact and h is a continuous map, there exist $\epsilon, \bar{\epsilon} \in \mathbb{R}_{>0}$ such that $\Lambda_{\epsilon}^{\mathbf{r}} y \in S_{\gamma_1}$, and $h(x) \neq 0$ for all $x \in C_{\gamma_1}$, where $C_{\gamma_1} = S_{\gamma_1} \cap \{x \in \mathbb{R}^n : |\Lambda_{\epsilon}^{\mathbf{r}} y - x| < \bar{\epsilon}\}$. Hence, for each $\gamma_2 \in (0, \gamma_0)$, there exist $\epsilon_{\gamma_2} \in \mathbb{R}_{>0}$ and an open set E_{γ_2} such that $E_{\gamma_2} \subset \bigcup_{\gamma_1 \in (0, \gamma_2)} C_{\gamma_1}$, $x = 0$ is in the boundary ∂E_{γ_2} of E_{γ_2}, $\partial E_{\gamma_2} \cap S_{\gamma_0}$ is nonempty and $\Lambda_{\epsilon}^{\mathbf{r}} y \in E_{\gamma_2}$ for all $\epsilon \in (0, \epsilon_{\gamma_2})$. Thus, $h(x) \neq 0$ for all $x \in E_{\gamma_2}$. Therefore $V(h(x)) > 0$ for all $x \in E_{\gamma_2}$.

Second, $\Delta V(\Lambda_{\epsilon}^{\mathbf{r}} y) = \Delta_h V(\Lambda_{\epsilon}^{\mathbf{r}} y) + V_d(\Lambda_{\epsilon}^{\mathbf{r}} y)$ is analysed along (1), where V_d is as defined in Appendix B and $\Delta_h V(\Lambda_{\epsilon}^{\mathbf{r}} y) = V(h(\Lambda_{\epsilon}^{\mathbf{r}} y)) - V(\Lambda_{\epsilon}^{\mathbf{r}} y)$. Note that $\Delta_h V(\Lambda_{\epsilon}^{\mathbf{r}} y) = \epsilon^{\nu m} V(h(y)) - \epsilon^m V(y) = \epsilon^{\nu m}[V(h(y)) - \epsilon^{(1-\nu)m} V(y)]$, thus,

$$\Delta V(\Lambda_{\epsilon}^{\mathbf{r}} y) \geq \epsilon^{\nu m}\left(V(h(y)) - \epsilon^{(1-\nu)m} V(y)\right) - |V_d(\Lambda_{\epsilon}^{\mathbf{r}} y)|.$$

But, since $E_{\gamma_2} \cup \partial E_{\gamma_2}$ is compact and the maps f and h are continuous, $\alpha-$Hölder continuity of V ensures the existence of $L_{E_{\gamma_2}} \in \mathbb{R}_{>0}$ such that $|V_d(\Lambda_{\epsilon}^{\mathbf{r}} y)| \leq L_{E_{\gamma_2}} |\Lambda_{\epsilon}^{-\nu\mathbf{r}} f(\Lambda_{\epsilon}^{\mathbf{r}} y) - h(y)|^{\alpha}$. Moreover, f is $D_{\mathbf{r}}-$homogeneous of degree $\nu \in (0,1)$ in the $(0, h)-$limit, then $\Lambda_{\epsilon}^{-\nu\mathbf{r}} f(\Lambda_{\epsilon}^{\mathbf{r}} y) - h(y) \to 0$ as $\epsilon \to 0$. Hence, there exist $\epsilon^*, \gamma^*(\epsilon^*) \in \mathbb{R}_{>0}$ such that for any $\gamma \in (0, \gamma^*)$ there is a point $x \in \mathcal{I}(V, \gamma) \setminus \{0\}$ such that $\Delta V(x) > 0$. Thus, from (Agarwal, 2000, Theorem 5.9.3) we conclude that $x = 0$ is an unstable equilibrium point of (1).

Appendix D. PROOF OF LEMMA 14

(1) The proof consists in verifying that $\Lambda_{\epsilon}^{\frac{1}{\nu}\mathbf{r}} h^{-1}(y) = h^{-1}(\Lambda_{\epsilon}^{\mathbf{r}} y)$. Since h is $D_{\mathbf{r}}-$homogeneous of degree ν, $h(\Lambda_{\delta}^{\mathbf{r}} x) = \Lambda_{\delta}^{\nu\mathbf{r}} h(x)$. Hence, by defining $y = h(x)$, we have that $\Lambda_{\delta}^{\mathbf{r}} x = h^{-1}(\Lambda_{\delta}^{\nu\mathbf{r}} y)$ and $\Lambda_{\delta}^{\mathbf{r}} x = \Lambda_{\delta}^{\mathbf{r}} h^{-1}(y)$. Thus, by defining $\epsilon = \delta^{\nu}$, we obtain $\Lambda_{\epsilon}^{\frac{1}{\nu}\mathbf{r}} h^{-1}(y) = h^{-1}(\Lambda_{\epsilon}^{\mathbf{r}} y)$.

(2) First, we consider the case of g being $D_{\mathbf{r}}-$homogeneous in the $(+\infty, h)-$limit. We have to prove that

$$\lim_{\epsilon \to +\infty} \left(\Lambda_{\epsilon}^{-\nu\mathbf{r}} g^{-1}(\Lambda_{\epsilon}^{\mathbf{r}} x) - h^{-1}(x)\right) = 0. \tag{D.1}$$

Define the functions $g_n : \mathbb{R}^n \to \mathbb{R}^n$ given by $g_n(x) = \Lambda_n^{-\nu\mathbf{r}} g(\Lambda_n^{\mathbf{r}} x)$. Since g is invertible and the diagonal matrix $\Lambda_n^{\mathbf{r}}$ is invertible for any $n \in \mathbb{Z}_{>0}$, the functions g_n are also invertible. Indeed, by denoting $z = g_n(x)$, we can see from the definition of g_n that $x = \Lambda_n^{-\mathbf{r}} g^{-1}(\Lambda_n^{\nu\mathbf{r}} z)$, therefore, $g_n^{-1}(z) = \Lambda_n^{-\mathbf{r}} g^{-1}(\Lambda_n^{\nu\mathbf{r}} z)$. Thus, by denoting $\epsilon = n^{\nu}$ we have that $g_n^{-1}(z) = \Lambda_{\epsilon}^{-\mathbf{r}/\nu} g^{-1}(\Lambda_{\epsilon}^{\mathbf{r}} z)$. Hence, it is clear that to verify (D.1) it is sufficient to prove that $g_n^{-1} \to h^{-1}$ uniformly as $n \to \infty$.

By hypothesis we know that $g_n \to h$ uniformly as $n \to \infty$. Now, under composition with a uniformly continuous function, a convergent sequence preserves the uniform convergence. Note that since h is a continuous map, h is uniformly continuous in any compact set. Thus, instead of proving that $g_n^{-1} \to h^{-1}$ we will prove that $h \circ g_n^{-1} \to h \circ h^{-1}$.

Note that $|h(g_n^{-1}(x)) - h(h^{-1}(x))| = |h(g_n^{-1}(x)) - x| = |h(g_n^{-1}(x)) - g_n(g_n^{-1}(x))|$. Since $g_n \to h$, we have that $\lim_{n \to \infty} |h(g_n^{-1}(x)) - g_n(g_n^{-1}(x))| = 0$. Hence, $g_n^{-1} \to h^{-1}$ as $n \to \infty$.

For the case of the $(0, h)-$limit, the proof is analogous but by defining the functions $g_n : \mathbb{R}^n \to \mathbb{R}^n$ given by $g_n(x) = \Lambda_{1/n}^{-\nu\mathbf{r}} g(\Lambda_{1/n}^{\mathbf{r}} x)$.